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An inverse scattering transform for the Landau–Lifschitz equation for a spin chain with an easy axis

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Abstract. An inverse scattering transform is developed to solve the Landau–Lifschitz equation for a spin chain with an easy axis. Instead of introducing the Riemann surface, which is required for double-valued functions of the usual spectral parameter λ, the transform is developed by making use of an auxiliary parameter ζ. The Marchenko equation, soliton solutions and asymptotic behaviours are derived. The results reduce naturally to those for the isotropic chain when the anisotropy vanishes.

1. Introduction

The equation of a continuous one-dimensional ferromagnet, known as the Landau–Lifschitz equation (Landau and Lifschitz 1935), has attracted much attention in the past two decades. A complete study for the isotropic case was first given by means of the inverse scattering transform (Laksmanan 1977, Takhtajan 1977, Fogedby 1980). The gauge isomorphism between it and the non-linear Schrödinger equation was demonstrated by Zakharov and Takhtajan (1979).

For the case of complete anisotropy, its Lax pair was found by Sklyanin (1979); it was then shown that the problem can be reduced to the Riemann boundary value problem on a torus, and was studied by using the method of the Riemann problem (Michailov 1980, 1982, Rodin 1983, 1984). However, the derivation and results are complicated and are expressed in terms of elliptic functions; the final results, such as the expression of the one-soliton solution, have never been obtained in an explicit manner. The problem can therefore by no means be considered as complete.

For anisotropy of the easy axis type, it is difficult to obtain the one-soliton solution by the method of integration after separating the variables (Tjon and Wright 1977). The problem was attacked by some authors (Bolovik 1978, Bolovik and Kulinich 1984) using the inverse scattering transform, but the study was not completed since the expressions of soliton solutions were not obtained.

An attempt was made to construct the exact multi-soliton solutions with the direct method of Hirota (Bogdan and Kovalev 1980). However, the authors were unable to prove a series of non-trivial identities on the parameters of the solution at the end. Explicit expressions for multi-solitons were not found. Recently, phase-shift analyses have been performed with the direct method of Hirota (Svendsen and Fogedby 1993).

The difficulty in using the inverse scattering transform to study the present problem lies in the complexity due to the Riemann surface, required for a double-valued function of

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the usual spectral parameter λ. To avoid this difficulty, an auxiliary parameter ζ (see (6) and (7)) is introduced. In terms of this parameter an inverse scattering transform procedure is performed without difficulty. The Marchenko equation is derived and soliton solutions are found by solving it in the reflectionless case. Asymptotic behaviours in the limits as $t \to \pm\infty$ are obtained as desired.

2. The Landau–Lifschitz equation for a spin chain with an easy axis

The Landau–Lifschitz (LL) equation for a spin chain with an easy axis is

$$S_t = S \times S_{xx} + S \times JS \qquad |S| = 1 \tag{1}$$

where the diagonal matrix J

$$J = \mathrm{diag}(0, 0, 16\rho^2) \tag{2}$$

characterizes the easy axis, the 3-axis. Here ρ is a real constant and the factor 16 is introduced for later convenience. The Lax pair of the equation is given by

$$L = -\mathrm{i}\kappa S_3\sigma_3 - \mathrm{i}\mu(S_1\sigma_1 + S_2\sigma_2) \tag{3}$$

$$M = \mathrm{i}2\mu^2 S_3\sigma_3 + \mathrm{i}2\mu\kappa(S_1\sigma_1 + S_2\sigma_2) - \mathrm{i}\mu(S_2S_{3x} - S_3S_{2x})\sigma_1 - \mathrm{i}\mu(S_3S_{1x} - S_1S_{3x})\sigma_2 - \mathrm{i}\kappa(S_1S_{2x} - S_2S_{1x})\sigma_3 \tag{4}$$

where parameters κ and μ satisfy

$$\mu^2 = \kappa^2 + 4\rho^2. \tag{5}$$

If one of these parameters is taken to be an independent parameter, then the other is a double-valued function of the first, and it is then necessary to introduce a Riemann surface. To avoid the complexity brought about by a Riemann surface, we introduce an auxiliary parameter ζ so that

$$\kappa = \zeta - \rho^2\zeta^{-1} \tag{6}$$

$$\mu = \zeta + \rho^2\zeta^{-1}. \tag{7}$$

The Lax equations are written as

$$\partial_x F(\zeta) = L(\zeta)F(\zeta) \tag{8}$$

$$\partial_t F(\zeta) = M(\zeta)F(\zeta). \tag{9}$$

We shall drop the arguments x and t henceforth, unless it is necessary to show them.

Since the 3-axis is an easy axis

$$S \to S_0 = (0, 0, 1) \qquad \text{as} \quad |x| \to \infty \tag{10}$$

we then have

$$\partial_x E(x, \zeta) = L_0(\zeta)E(x, \zeta) \tag{11}$$

where

$$L_0(\zeta) = -\mathrm{i}\kappa\sigma_3 \tag{12}$$

$$E(x,\zeta) = \mathrm{e}^{-\mathrm{i}\kappa x\sigma_3} \tag{13}$$

i.e.

$$E_{\cdot 1}(x,\zeta) = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \mathrm{e}^{-\mathrm{i}\kappa x} \qquad E_{\cdot 2}(x,\zeta) = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \mathrm{e}^{\mathrm{i}\kappa x}. \tag{14}$$

The Jost solution $\Psi(x,\zeta)$ of (8) is defined as

$$\Psi(x,\zeta) \to E(x,\zeta) \qquad \text{as} \quad x \to \infty. \tag{15}$$

By using standard procedures of the theory of characteristics, one can find the following integral representation:

$$\Psi(x,\zeta) = E(x,\zeta) + \kappa \int_x^\infty \mathrm{d}y K^{\mathrm{d}}(x,y)\, E(y,\zeta) + \mu \int_x^\infty \mathrm{d}y K^{\mathrm{nd}}(x,y)\, E(y,\zeta) \tag{16}$$

where the superscripts 'd' and 'nd' denote the diagonal and the non-diagonal parts of the matrix, respectively, and

$$K(x,+\infty) = 0. \tag{17}$$

In fact, the procedure is as follows. First, let us suppose that the representation (16) holds. Then substituting (16) into (8), and integrating by parts, one can obtain

$$\mathrm{i}\sigma_3 + K^{\mathrm{d}}(x,x) - \mathrm{i}S_3\sigma_3 - S_3\sigma_3 K^{\mathrm{d}}(x,x)\sigma_3 - (S_1\sigma_1 + S_2\sigma_2)K^{\mathrm{nd}}(x,x)\sigma_3 = 0 \tag{18}$$

$$K^{\mathrm{nd}}(x,x) - \mathrm{i}(S_1\sigma_1 + S_2\sigma_2) - S_3\sigma_3 K^{\mathrm{nd}}(x,x)\sigma_3 - (S_1\sigma_1 + S_2\sigma_2)K^{\mathrm{d}}(x,x)\sigma_3 = 0 \tag{19}$$

$$K_x^{\mathrm{nd}}(x,y)\sigma_3 + (S_1\sigma_1 + S_2\sigma_2)K_y^{\mathrm{d}}(x,y) + S_3\sigma_3 K_y^{\mathrm{nd}}(x,y) = 0 \qquad y = x \tag{20}$$

$$K_x^{\mathrm{d}}(x,y)\sigma_3 + (S_1\sigma_1 + S_2\sigma_2)K_y^{\mathrm{nd}}(x,y) + S_3\sigma_3 K_y^{\mathrm{d}}(x,y) = 0 \qquad y \geqslant x \tag{21}$$

$$K_{xy}^{\mathrm{nd}}(x,y)\sigma_3 + (S_1\sigma_1 + S_2\sigma_2)K_{yy}^{\mathrm{d}}(x,y) + S_3\sigma_3 K_{yy}^{\mathrm{nd}}(x,y) - 4\rho^2 K^{\mathrm{nd}}(x,y) = 0 \qquad y \geqslant x. \tag{22}$$

Equations (18)–(22) form a Coursal differential problem and can be transformed into a system of integral equations. From the theory of characteristics, there exists a unique solution of the equations (21) and (22) under the conditions (18)–(20). That is, the assumption (16) is valid. An important relation follows:

$$(S\cdot\sigma) = \big(I - \mathrm{i}K(x,x)\sigma_3\big)\sigma_3\big(I - \mathrm{i}K(x,x)\sigma_3\big)^{-1}. \tag{23}$$

Similarly, the Jost solution $\Phi(x,\zeta)$ of (8) is defined as

$$\Phi(x,\zeta) \to E(x,\zeta) \qquad \text{as} \quad x \to -\infty \tag{24}$$

and

$$\Phi(x,\zeta) = E(x,\zeta) + \kappa \int_{-\infty}^x \mathrm{d}y N^{\mathrm{d}}(x,y)\, E(y,\zeta) + \mu \int_{-\infty}^x \mathrm{d}y N^{\mathrm{nd}}(x,y)\, E(y,\zeta) \tag{25}$$

$$N(x,-\infty) = 0. \tag{26}$$

3. The properties of the Jost solutions

In the plane of complex ζ, one can see the correspondence

$$\text{real } \zeta \leftrightarrow \text{real } \kappa \qquad \operatorname{Im}\zeta > 0 \leftrightarrow \operatorname{Im}\kappa > 0 \qquad \operatorname{Im}\zeta < 0 \leftrightarrow \operatorname{Im}\kappa < 0. \tag{27}$$

Hence, with the help of (16) and (20), analyticities of the Jost solutions $\Psi(\zeta)$ and $\Phi(\zeta)$ can be simply derived and the results are similar to those in the case of the NLS equation. We write the expressions for real ζ

$$\Psi(x,\zeta) = \left(\tilde{\psi}(x,\zeta)\ \psi(\zeta)\right) \qquad \Phi(x,\zeta) = \left(\psi(x,\zeta),\ \tilde{\psi}(x,\zeta)\right) \tag{28}$$

$$\Phi(x,\zeta) = \Psi(x,\zeta)T(\zeta) \tag{29}$$

where

$$T(\zeta) = \begin{pmatrix} a(\zeta) & -\tilde{b}(\zeta) \\ b(\zeta) & \tilde{a}(\zeta) \end{pmatrix}. \tag{30}$$

From (29) it follows that

$$\phi(\zeta) = a(\zeta)\tilde{\psi}(\zeta) + b(\zeta)\psi(\zeta) \tag{31}$$

for example.

The terms $\psi(\zeta)$, $\phi(\zeta)$ and $a(\zeta)$ can be analytically continued into the upper-half plane of complex ζ, while $\tilde{\psi}(\zeta)$, $\tilde{\phi}(\zeta)$ and $\tilde{a}(\zeta)$ the lower-half plane of complex ζ. Furthermore we have

$$\tilde{\psi}(\zeta) = \mathrm{i}\sigma_2\overline{\psi(x,\bar{\zeta})} \qquad \tilde{\phi}(\zeta) = -\mathrm{i}\sigma_2\overline{\phi(x,\bar{\zeta})} \tag{32}$$

$$\tilde{a}(\zeta) = \overline{a(\bar{\zeta})}. \tag{33}$$

However, $b(\zeta)$ and $\tilde{b}(\zeta)$ cannot be analytically continued out of the real axis of ζ, in general. It has the following property:

$$\tilde{b}(\zeta) = \overline{b(\zeta)} \tag{34}$$

for real ζ.

In the upper-half plane of ζ, $a(\zeta)$ may have zeros. Suppose the zero points of $a(\zeta)$ are at $\zeta_1, \zeta_2, \cdots$; one has

$$\phi(\zeta_n) = b_n\psi(\zeta_n). \tag{35}$$

From (33), in the lower-half plane of ζ, $\tilde{a(\zeta)}$ has zeros, $\bar{\zeta}_1, \bar{\zeta}_2, \cdots$, and

$$\tilde{\phi}(\bar{\zeta}_n) = -\bar{b}_n\tilde{\psi}(\bar{\zeta}_n). \tag{36}$$

4. The reduction properties of the Jost solutions

From (3) and (4)

$$L(-\rho^2\zeta^{-1}) = \sigma_3 L(\zeta)\sigma_3 \qquad M(-\rho^2\zeta^{-1}) = \sigma_3 M(\zeta)\sigma_3 \tag{37}$$

and (13)

$$F_0(-\rho^2\zeta^{-1}) = \sigma_3 F_0(\zeta)\sigma_3 \tag{38}$$

one can obtain

$$\tilde{\psi}(x, -\rho^2\zeta^{-1}) = \sigma_3\tilde{\psi}(x,\zeta) \qquad \psi(x, -\rho^2\zeta^{-1}) = -\sigma_3\psi(x,\zeta) \tag{39}$$

$$\phi(x, -\rho^2\zeta^{-1}) = \sigma_3\phi(x,\zeta) \qquad \tilde{\phi}(x, -\rho^2\zeta^{-1}) = -\sigma_3\tilde{\phi}(x,\zeta) \tag{40}$$

and

$$a(-\rho^2\zeta^{-1}) = a(\zeta) \tag{41}$$

$$b(-\rho^2\zeta^{-1}) = b(\zeta) \tag{42}$$

for real ζ.

Under the transformation

$$\zeta \to -\rho^2\zeta^{-1} \tag{43}$$

the Jost solutions, $a(\zeta)$, leave κ invariant; this property is called the reduction transformation property.

From (41), if ζ_n is a zero point of $a(\zeta)$, then $-\rho^2\zeta_n^{-1}$ is also a zero point of $a(\zeta)$. We write

$$\zeta_{N+n} = -\rho^2\zeta_n^{-1} \tag{44}$$

then from (39) and (40) it follows that

$$b_{N+n} = -b_n. \tag{45}$$

Integrating by parts, from (15) one can obtain

$$\psi(x,\zeta) = \mathrm{O}(1) \qquad \text{as } |\zeta| \to \infty. \tag{46}$$

Similarly, from (21) we have

$$\phi(x,\zeta) = \mathrm{O}(1) \qquad \text{as } |\zeta| \to \infty. \tag{47}$$

Hence we have

$$\lim_{|\zeta|\to\infty} a(\zeta) = a_0(\text{constant}). \tag{48}$$

With the aid of (39) and (40) we also see

$$\lim_{|\zeta|\to 0} a(\zeta) = a_0(\text{constant}). \tag{49}$$

Suppose $a(\zeta)$ has N pairs of zeros, $\zeta_1, \zeta_2, \cdots, \zeta_{2N}$, labelled according to (44). The factor of $a(\zeta)$ due to these zeros, $a_d(\zeta)$ is

$$a_d(\zeta) = \prod_{n=1}^{2N} \frac{\zeta - \zeta_n}{\zeta - \bar{\zeta}_n} = \prod_{n=1}^{N} \frac{\zeta - \zeta_n}{\zeta - \bar{\zeta}_n} \frac{\zeta + \rho^2 \zeta_n^{-1}}{\zeta + \rho^2 \bar{\zeta}_n^{-1}} \frac{\bar{\zeta}_n - \rho^2 \bar{\zeta}_n^{-1}}{\zeta_n - \rho^2 \zeta_n^{-1}}. \tag{50}$$

The factor of $a(\zeta)$ due to the continuous spectrum, $a_c(\zeta)$ is

$$a_c(\zeta) = \exp\left(\frac{1}{2\pi \mathrm{i}} \int_{-\infty}^{\infty} \mathrm{d}\zeta' \frac{\zeta' + \rho^2 \zeta'^{-1}}{(\zeta' - \zeta)(\zeta' + \rho^2 \zeta^{-1})} \ln |a_c(\zeta')|^2\right). \tag{51}$$

When ζ tends to the real axis, we take $\zeta + \mathrm{i}0$ as ζ in the integral.

We then obtain

$$a(\zeta) = a_d(\zeta) a_c(\zeta). \tag{52}$$

In the limit $\rho \to 0$, $a(\zeta)$ reduces to the corresponding expression in the case of an isotropic chain.

It is obvious that

$$\lim_{|\zeta|\to\infty} |a(\zeta)|^2 = \lim_{|\zeta|\to 0} |a(\zeta)|^2 = 1. \tag{53}$$

Hence the integral in (41) is equal to the residues at ζ and $-\rho^2\zeta^{-1}$. When $\zeta = \rho$ or $-\rho$, the integral vanishes, on account of (40). Therefore, we have

$$a(\rho) = a(-\rho) = 1. \tag{54}$$

5. The Marchenko equation

From (32) we have

$$a(\zeta)^{-1}\phi(x, \zeta) - E_{.1}(x, \zeta) = \tilde{\psi}(x, \zeta) - E_{.1}(x, \zeta) + r(\zeta)\psi(x, \zeta) \tag{55}$$

where $r(\zeta) = b(\zeta)/a(\zeta)$. Multiplying it by $(2\pi\kappa)^{-1} \exp(\mathrm{i}\kappa x)$ in the case of $y > x$, and integrating with respect to ζ along a path Γ which is the real axis from $-\infty$ to ∞, except in the neighbourhood of ρ and $-\rho$ where it is replaced by two small semi-circles in the lower half plane, we obtain the Marchenko equation:

$$\begin{pmatrix} K_{11}(x, y) \\ K_{21}(x, y) \end{pmatrix} + \begin{pmatrix} 0 \\ F(x + y) \end{pmatrix} + \int_x^{\infty} \mathrm{d}z \begin{pmatrix} K_{12}(x, z) F''(z + y) \\ K_{22}(x, z) F'(z + y) \end{pmatrix} = 0 \tag{56}$$

where $y > x$ and

$$F(x) = \sum_{n=1}^{2N} c_n \kappa_n^{-1} e^{i\kappa_n x} + \frac{1}{2\pi} \int d\zeta\, r(\zeta) \kappa^{-1} e^{i\kappa x} \tag{57}$$

$$F'(x) = \sum_{n=1}^{2N} c_n e^{i\kappa_n x} + \frac{1}{2\pi} \int d\zeta\, r(\zeta) e^{i\kappa x} \tag{58}$$

$$F''(x) = \sum_{n=1}^{2N} c_n \kappa_n^{-1} \mu_n e^{i\kappa_n x} + \frac{1}{2\pi} \int d\zeta\, r(\zeta) \kappa^{-1} \mu e^{i\kappa x} \tag{59}$$

$$c_n = \frac{b_n}{i\dot{a}(\zeta_n)} \tag{60}$$

$$\dot{a}(\zeta_n) = \frac{d}{d\zeta} a(\zeta)|_{\zeta=\zeta_n}. \tag{61}$$

Here we have used

$$\int d\zeta e^{i\kappa x} = 2\pi \delta(x) \tag{62}$$

$$\int d\zeta\, \zeta^{-1} f(\zeta) e^{i\kappa x} = 0 \qquad f(-\rho^2 \zeta^{-1}) = f(\zeta). \tag{63}$$

We notice that on the left-hand side the contributions due to poles at ρ and $-\rho$ cancel each other out, due to (54).

With the property

$$K_{22}(x, y) = \overline{K_{11}(x, y)} \qquad K_{12}(x, y) = -\overline{K_{21}(x, y)} \tag{64}$$

we obtain from (56)

$$\begin{pmatrix} K_{12}(x, y) \\ K_{22}(x, y) \end{pmatrix} + \begin{pmatrix} -\overline{F(x + y)} \\ 0 \end{pmatrix} - \int_x^\infty dz \begin{pmatrix} K_{11}(x, z)\overline{F'(z + y)} \\ K_{21}(x, z)\overline{F''(z + y)} \end{pmatrix} = 0. \tag{65}$$

Equations (56) and (65) are the desired Marchenko equation.

The time dependence can be obtained by standard procedures from the second Lax equation (9). It can be achieved by the following replacements:

$$b(\zeta) \to b(\zeta, t) = b(\zeta, 0) e^{-i4\mu^2 t} \tag{66}$$

$$b_n \to b_n(t) = b_n(0) e^{-i4\mu_n^2 t} \tag{67}$$

$$a(\zeta) \to a(\zeta, t) = a(\zeta, 0) \tag{68}$$

where $b(\zeta, 0)$ etc, are constants.

6. The reflectionless case

In the reflectionless case, $r(\zeta) = 0$, the Marchenko equation can be considerably simplified. From (43) we have

$$\dot{a}(\zeta_{N+n}) = \rho^{-2}\zeta_n^2\dot{a}(\zeta_n) \qquad n = 1, 2, \cdots, N. \tag{69}$$

Hence we obtain

$$c_{N+n} = -\rho^2\zeta_n^{-2}c_n. \tag{70}$$

Noticing that the transformation (42) leaves κ unchanged, we have

$$\kappa_{N+n} = \kappa_n \tag{71}$$

$$\mu_{N+n} = -\mu_n. \tag{72}$$

Taking these three formulae into consideration, from (57)–(59) we have in the reflectionless case

$$F(x) = \sum_{n=1}^{N}(1 - \rho^2\zeta_n^{-2})\kappa_n^{-1}e^{i\kappa_n x} = \sum_{n=1}^{N}\zeta_n^{-1}e^{i\kappa_n x} \tag{73}$$

$$F'(x) = \sum_{n=1}^{N}(1 - \rho^2\zeta_n^{-2})e^{i\kappa_n x} = \sum_{n=1}^{N}\zeta_n^{-1}\kappa_n e^{i\kappa_n x} \tag{74}$$

$$F''(x) = \sum_{n=1}^{N}(1 + \rho^2\zeta_n^{-2})\kappa_n^{-1}\mu_n e^{i\kappa_n x} = \sum_{n=1}^{N}(\zeta_n\kappa_n)^{-1}\mu_n^2 e^{i\kappa_n x}. \tag{75}$$

Consider the Marchenko equation

$$K_{11}(x, y) + \int_x^\infty dz\, K_{12}(x, z)F''(z + y) = 0 \tag{76}$$

$$K_{12}(x, y) - \overline{F(x + y)} - \int_x^\infty dz\, K_{11}(x, z)\overline{F'(z + y)} = 0. \tag{77}$$

We now write them in matrix form

$$F(x + y) = C(x)D(y)^{\mathrm{T}} \tag{78}$$

$$F'(x + y) = C'(x)D(y)^{\mathrm{T}} \qquad F''(x + y) = C''(x)D(y)^{\mathrm{T}} \tag{79}$$

where

$$C(x)_n = \zeta_n^{-1}c_n D(x)_n \tag{80}$$

$$C'(x)_n = \zeta_n^{-1}\kappa_n c_n D(x)_n \qquad C''(x)_n = (\zeta_n\kappa_n)^{-1}\mu_n^2 c_n D(x)_n \tag{81}$$

$$D(x)_n = e^{i\kappa_n x}. \tag{82}$$

Writing

$$K_{11}(x,y) = K_{11}(x)D(y)^{\mathrm{T}} \qquad K_{12}(x,y) = K_{12}(x)\overline{D(y)}^{\mathrm{T}} \tag{83}$$

equations (76) and (77) are now written in matrix form:

$$K_{11}(x) + K_{12}(x)A''(x) = 0 \tag{84}$$

$$K_{12}(x) - \overline{C(x)} - K_{11}(x)A'(x) = 0 \tag{85}$$

where $A'(x)$ and $A''(x)$ are $N \times N$ matrices:

$$A''(x)_{nm} = \frac{1}{\mathrm{i}(\bar{\kappa}_n - \kappa_m)}\overline{D(x)}_n C''(x)_m \tag{86}$$

$$A'(x)_{nm} = \frac{1}{-\mathrm{i}(\kappa_n - \bar{\kappa}_m)} D(x)_n \overline{C'(x)}_m. \tag{87}$$

It can be found that

$$K_{12}(x) = \overline{C(x)}\big(I + A''(x)A'(x)\big)^{-1} \tag{88}$$

$$K_{11}(x) = -\overline{C(x)}\big(I + A''(x)A'(x)\big)^{-1}A''(x). \tag{89}$$

Hence we obtain

$$\begin{aligned} K_{12}(x,x) &= \overline{C}(I + A''A')^{-1}\overline{D}^{\mathrm{T}} = \mathrm{Tr}\big[\overline{D}^{\mathrm{T}}\overline{C}(I + A''A')^{-1}\big] \\ &= \frac{\det(I + A''A' + \overline{D}^{\mathrm{T}}\overline{C})}{\det(I + A''A')} - 1 \end{aligned} \tag{90}$$

$$\begin{aligned} 1 - \mathrm{i}K_{11}(x,x) &= 1 + \mathrm{i}\overline{C}(I + A''A')^{-1}A''D^{\mathrm{T}} = 1 + \mathrm{Tr}\big[\mathrm{i}A''D^{\mathrm{T}}\overline{C}(I + A''A')^{-1}\big] \\ &= \frac{\det(I + A''B')}{\det(I + A''A')} \end{aligned} \tag{91}$$

where

$$B'(x) = A'(x) + \mathrm{i}D(x)^{\mathrm{T}}\overline{C(x)} \tag{92}$$

or

$$B'_{nm}(x) = \frac{1}{-\mathrm{i}(\kappa_n - \bar{\kappa}_m)}\kappa_n D_n(x)\bar{\kappa}_m^{-1}\overline{C'(x)}_m. \tag{93}$$

From (17) we obtain

$$S_3 = \Xi^{-1}[(1 - \mathrm{i}K_{11})(1 + iK_{22}) + K_{12}K_{21}] = \Xi^{-1}(|1 - \mathrm{i}K_{11}|^2 - |K_{12}|^2) \tag{94}$$

$$S_1 - \mathrm{i}S_2 = -\Xi^{-1}[2(1 - \mathrm{i}K_{11})K_{12}] \tag{95}$$

where

$$\Xi = \det(I - \mathrm{i}K\sigma_3) = |1 - \mathrm{i}K_{11}|^2 + |K_{12}|^2. \tag{96}$$

Setting $S_3 = \cos\theta$ and $S_1 - \mathrm{i}S_2 = \sin\theta\,\mathrm{e}^{-\mathrm{i}\phi}$, we obtain expressions for θ:

$$\cos\theta = 1 - 2\frac{|K_{12}|^2}{|1 - \mathrm{i}K_{11}|^2 - |K_{12}|^2}. \tag{97}$$

The expression for ϕ can be written as

$$\phi = -\arg K_{12} - \arg(1 - \mathrm{i}K_{11}). \tag{98}$$

7. The one-soliton solution

When $N = 1$, from (90) and (91) we have

$$K_{12}(x,x) = \Delta^{-1}\overline{D}_1^2\bar{c}_1\bar{\zeta}_1^{-1} \tag{99}$$

$$1 - \mathrm{i}K_{11}(x,x) = \Delta^{-1}(1 + |D_1|^4|c_1|^2\mu_1^2|\zeta_1|^{-2}|\kappa_1 - \bar{\kappa}_1|^{-2}) \tag{100}$$

$$\Delta = 1 + |D_1|^4|c_1|^2\mu_1^2|\zeta_1|^{-2}|\kappa_1 - \bar{\kappa}_1|^{-2}\bar{\kappa}_1\kappa_1^{-1} \tag{101}$$

where

$$c_1 = c_{10}\mathrm{e}^{-\mathrm{i}4\mu_1^2 t} \tag{102}$$

and c_{10} is a constant.

Introduce

$$D_1^2 c_1\zeta_1^{-1}|\mu_1||\kappa_1 - \bar{\kappa}_1|^{-1} = \mathrm{e}^{-\Theta_1}\mathrm{e}^{\mathrm{i}\Phi_1} \tag{103}$$

where

$$\Theta_1 = 2\kappa_1''(x - 4\kappa_1' t - x_1) \tag{104}$$

$$\Phi_1 = 2\kappa_1' x - 4(\kappa_1'^2 - \kappa_1''^2 + 4\rho^2)t + \phi_{10} \tag{105}$$

$$\kappa_1 = \kappa_1' + \mathrm{i}\kappa_1''. \tag{106}$$

From (97) we obtain

$$\theta = 1 - 2\frac{\kappa_1''^2}{|\mu_1|^2}[\cosh^2\Theta_1 + \tfrac{1}{2}(4\rho^2 + |\kappa_1|^2 - |\mu_1|^2)|\mu_1|^{-2}]^{-1}. \tag{107}$$

Substituting into (98), we obtain

$$\phi = \Phi_1 + \tan^{-1}\left(\frac{\mu_1''}{\mu_1'}\tanh\Theta_1\right) + 2\tan^{-1}\left(\frac{\mu_1'\kappa_1'' - \mu_1''\kappa_1'}{\mu_1'\kappa_1' + \mu_1''\kappa_1''}\tanh\Theta_1\right). \tag{108}$$

When $\rho \to 0$, we have

$$\zeta_1 = \kappa_1 = \mu_1 \tag{109}$$

equations (107) and (108) tend obviously to those of the isotropic chain.

8. Explicit expressions of multi-soliton solutions

From equation (A3) in the appendix, it is convenient to express $\dot{a}(\zeta_n)$ in terms of κ. Equation (49) is expressed as

$$a(\kappa) = \prod_{n=1}^{N} \frac{\kappa - \kappa_n}{\kappa - \bar{\kappa}_n} \frac{\bar{\kappa}_n}{\kappa_n}. \tag{110}$$

Hence we have

$$\dot{a}(\kappa_n) = \frac{\mathrm{d}}{\mathrm{d}\kappa} a(\kappa)\big|_{\kappa=\kappa_n} = \left(\frac{\mathrm{d}\kappa}{\mathrm{d}\zeta}\right)^{-1} \frac{\mathrm{d}}{\mathrm{d}\zeta} a(\zeta)\big|_{\zeta=\zeta_n} = \frac{\zeta_n}{\kappa_n} \dot{a}(\zeta_n) \tag{111}$$

and

$$\dot{a}(\kappa_n) = \prod_{m\neq n} \frac{\kappa_n - \kappa_m}{\kappa_n - \bar{\kappa}_m} \frac{1}{\kappa_n - \bar{\kappa}_n} \prod_{l=1}^{N} \frac{\bar{\kappa}_l}{\kappa_l}. \tag{112}$$

Therefore, we have

$$c_n = \frac{\zeta_n}{\kappa_n} \frac{b_n}{\mathrm{i}\dot{a}(\kappa_n)}. \tag{113}$$

With this notation, we write

$$\Delta \equiv \det(I + A''A') \tag{114}$$

and then (114) can be expressed as

$$\Delta = 1 + \sum_{r=1}^{N} \sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N} \sum_{1\leqslant m_1<m_2<\cdots<m_r\leqslant N} \Delta(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) \tag{115}$$

where

$$\Delta(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) = (-1)^r \prod_n \prod_m \bar{f}_n \bar{\alpha}_n \bar{\kappa}_n \bar{\mu}_n^{-1} f_m \alpha_m \mu_m \kappa_m^{-1} (\bar{\kappa}_n - \kappa_m)^{-2}$$
$$\times \prod_{n<n'} \prod_{m<m'} (\bar{\kappa}_n - \bar{\kappa}_{n'})^2 (\kappa_m - \kappa_{m'})^2 \tag{116}$$

and

$$\alpha_n = \prod_{m\neq n} \frac{\kappa_n - \kappa_m}{\kappa_n - \bar{\kappa}_m} \frac{1}{\kappa_n - \bar{\kappa}_n} \tag{117}$$

$$f_n = D_n^2 b_n \prod_{l=1}^{N} \frac{\bar{\kappa}_l}{\kappa_l}. \tag{118}$$

Equation (118) can be expressed as

$$f_n = \mathrm{e}^{\mathrm{i}\Phi_n} \mathrm{e}^{-\Theta_n} \tag{119}$$

where

$$\Theta_n = 2\kappa_n''(x - 4\kappa_n' t - x_n) \tag{120}$$

$$\Phi_n = 2\kappa_n' x - 4(\kappa_n'^2 - \kappa_n''^2 + 4\rho^2)t + \Phi_{n0} \tag{121}$$

$$\kappa_n = \kappa_n' + \mathrm{i}\kappa_n'' \qquad \kappa_n'' > 0 \tag{122}$$

and x_n and Φ_{n0} are real constants.

Similarly, we have

$$\Delta_{11} \equiv \det(I + A''B') \tag{123}$$

$$\Delta_{11} = 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} \sum_{1 \leqslant m_1 < m_2 < \cdots < m_r \leqslant N} \Delta_{11}(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) \tag{124}$$

and

$$\Delta_{11}(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) = (-1)^r \prod_n \prod_m \bar{f}_n \overline{\alpha}_n \bar{\mu}_n^{-1} f_m \alpha_m \mu_m (\bar{\kappa}_n - \kappa_m)^{-2} \times \prod_{n<n'} \prod_{m<m'} (\bar{\kappa}_n - \bar{\kappa}_{n'})^2 (\kappa_m - \kappa_{m'})^2. \tag{125}$$

Similarly, we also have

$$\Delta_{12} \equiv \det(I + Q''Q') - \det(I + A''A') \tag{126}$$

$$\Delta_{12} = \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} \sum_{1 \leqslant m_2 < \cdots < m_r \leqslant N} \Delta_{12}(n_1, n_2, \cdots, n_r; 0, m_2, \cdots, m_r) \tag{127}$$

where

$$\Delta_{12}(n_1, n_2, \cdots, n_r; 0, m_2, \cdots, m_r) = (-1)^{r+1} \prod_n \prod_m \bar{f}_n \overline{\alpha}_n \bar{\mu}_n^{-1} f_m \alpha_m \mu_m (\bar{\kappa}_n - \kappa_m)^{-2} \times \prod_{n<n'} \prod_{m<m'} (\bar{\kappa}_n - \bar{\kappa}_{n'})^2 (\kappa_m - \kappa_{m'})^2 \tag{128}$$

where n, n' and m, m' satisfy (A11).

Writing

$$G = I - \mathrm{i}K\sigma_3 \tag{129}$$

we then have

$$G_{11} = \Delta^{-1}\Delta_{11} \qquad G_{12} = \Delta^{-1}\Delta_{12}. \tag{130}$$

Hence we have

$$\cos\theta = 1 - 2\frac{|\Delta_{12}|^2}{|\Delta_{11}|^2 + |\Delta_{12}|^2} \tag{131}$$

and

$$\phi = -\arg\Delta_{12} - \arg\Delta_{11}. \tag{132}$$

It can be seen that the formulae reduce to those for the anisotropic chain when ρ vanishes, as we have seen, in the one-soliton case.

9. Asymptotic behaviours in the limit $t \to \pm\infty$

Since all κ_n'' are positive, we suppose

$$\kappa_1' > \kappa_2' > \cdots > \kappa_N'. \tag{133}$$

The vicinity of $x = x_n + 4\kappa_n' t$ will be denoted by Ω_n. In the limit $t \to \infty$ these vicinities must separate, from left to right, as

$$\Omega_N \ \Omega_{N-1} \ \cdots \ \Omega_1. \tag{134}$$

In the vicinity Ω_j, we have

$$x - x_n - 4\kappa_n' t \to -\infty \qquad |f_n| \to \infty \quad n < j \tag{135}$$

$$x - x_m - 4\kappa_m' t \to +\infty \qquad |f_n| \to 0 \quad m > j. \tag{136}$$

Therefore, in this case, we have

$$\Delta \sim \Delta(1, 2, \cdots, j-1; 1, 2, \cdots, j-1) + \Delta(1, 2, \cdots, j; 1, 2, \cdots, j) \tag{137}$$

$$\Delta_{11} \sim \Delta_{11}(1, 2, \cdots, j-1; 1, 2, \cdots, j-1) + \Delta_{11}(1, 2, \cdots, j; 1, 2, \cdots, j) \tag{138}$$

$$\Delta_{12} \sim \Delta_{12}(1, 2, \cdots, j; 0, 1, \cdots, j-1) \tag{139}$$

since we retain the terms proportional to $|f_1|^2|f_2|^2 \cdots |f_{j-1}|^2$. Substituting the explicit expressions (116), (123) and (128), except for a common factor, they become

$$\Delta \sim 1 + \left|f_j^{(+)}\right|^2 \bar{\kappa}_j \kappa_j^{-1} \mu_j \bar{\mu}_j^{-1} \tag{140}$$

$$\Delta_{11} \sim 1 + \left|f_j^{(+)}\right|^2 \mu_j \bar{\mu}_j^{-1} \tag{141}$$

$$\Delta_{12} \sim 2\kappa_j'' \overline{f_j^{(+)}} \, \bar{\mu}_j^{-1} \tag{142}$$

where

$$f_j^{(+)} = f_j \prod_{n=1}^{j-1} \frac{\kappa_j - \kappa_n}{\kappa_j - \bar{\kappa}_n} \prod_{m=j+1}^{N} \frac{\kappa_j - \bar{\kappa}_m}{\kappa_j - \kappa_m}. \tag{143}$$

Therefore, when $t \to \infty$, the N-soliton solution in the vicinity Ω_j, is approximately equal to the one-soliton solution with f_j replaced by $f_j^{(+)}$. The additional displacement of the centre is

$$x_j^{(+)} = \frac{1}{2\kappa_j''}\left(\ln \prod_{n=1}^{j-1}\left|\frac{\kappa_j - \kappa_n}{\kappa_j - \bar{\kappa}_n}\right| - \ln \prod_{m=j+1}^{N}\left|\frac{\kappa_j - \kappa_m}{\kappa_j - \bar{\kappa}_m}\right|\right) \tag{144}$$

and the additional phase shift is

$$\Phi_{j0}^{(+)} = \arg\left(\prod_{n=1}^{j-1} \frac{\kappa_j - \kappa_n}{\kappa_j - \bar{\kappa}_n}\right) - \arg\left(\prod_{m=j+1}^{N} \frac{\kappa_j - \kappa_m}{\kappa_j - \bar{\kappa}_m}\right). \tag{145}$$

Similarly, when $t \to -\infty$, in the vicinity Ω_j, the approximate N-soliton solution can be obtained from the above formulae by the simple replacements

$$x_j^{(+)} \to x_j^{(-)} = -x_j^{(+)} \qquad \Phi_{j0}^{(+)} \to \Phi_{j0}^{(-)} = -\Phi_{j0}^{(+)}. \tag{146}$$

Therefore, the total displacement of the centre and the total phase shift of the jth peak in the course from $t \to -\infty$ to $t \to \infty$ are $2x_j^{(+)}$ and $2\Phi_{j0}^{(+)}$, respectively.

10. Conclusion

Though the Marchenko equation for the spin chain with an easy axis seems rather more complicated than that for the isotropic chain, the above calculational procedure considerably simplifies the final results. Equations (116), (125) and (128) have some factors, such as $\bar{\kappa}_n\bar{\mu}_n^{-1}\kappa_m^{-1}\mu_m$, which appear in the corresponding formulae for the isotropic chain, in addition to the different time dependence (66)–(68). Therefore, the work to solve the Landau–Lifschitz equation for a spin chain with an easy axis is now complete.

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Appendix

By means of the Binet–Cauchy formula, we obtain

$$\det(I + A''A') = 1 + \sum_{r=1}^{N} \sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N} \sum_{1\leqslant m_1<m_2<\cdots<m_r\leqslant N} \times A''(n_1,n_2,\cdots,n_r;m_1,m_2,\cdots,m_r)A'(m_1,m_2,\cdots,m_r;n_1,n_2,\cdots,n_r) \tag{A1}$$

where $A''(n_1,n_2,\cdots,n_r;m_1,m_2,\cdots,m_r)$ denotes a minor that is a determinant of a submatrix of A'' by the remaining $(n_1,n_2,\cdots,n_r)$th rows and $(m_1,m_2,\cdots,m_r)$th columns. By using the well known formula from linear algebra

$$\det\big((x_j + x_k)^{-1}\big) = \prod_{j,k}(x_j + y_k)^{-1} \prod_{j<j'}\prod_{k<k'}(x_j - x_{j'})(y_k - y_{k'}) \tag{A2}$$

we obtain

$$A''(n_1,n_2,\cdots,n_r;m_1,m_2,\cdots,m_r)A'(m_1,m_2,\cdots,m_r;n_1,n_2,\cdots,n_r) = (-1)^r \times \prod_n\prod_m \overline{D}_n\overline{C'}_n D_m C''_m(\bar{\kappa}_n - \kappa_m)^{-2} \prod_{n<n'}\prod_{m<m'}(\bar{\kappa}_n - \bar{\kappa}_{n'})^2(\kappa_m - \kappa_{m'})^2 \tag{A3}$$

where

$$n, n' \in \{n_1, n_2, \cdots, n_r\} \qquad m, m' \in \{m_1, m_2, \cdots, m_r\}. \tag{A4}$$

Similarly, we have

$$\det(I + A''B') = 1 + \sum_{r=1}^{N} \sum_{1\leqslant n_1<n_2<\cdots<n_r\leqslant N} \sum_{1\leqslant m_1<m_2<\cdots<m_r\leqslant N} \times A''(n_1,n_2,\cdots,n_r;m_1,m_2,\cdots,m_r)B'(m_1,m_2,\cdots,m_r;n_1,n_2,\cdots,n_r) \tag{A5}$$

and

$$A''(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r)B'(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) = (-1)^r \times \prod_n \prod_m \overline{D}_n \overline{C'}_n D_m C''_m \bar{\kappa}_n^{-1} \kappa_m (\bar{\kappa}_n - \kappa_m)^{-2} \prod_{n<n'} \prod_{m<m'} (\bar{\kappa}_n - \bar{\kappa}_{n'})^2 (\kappa_m - \kappa_{m'})^2. \tag{A6}$$

To calculate $\det(I + A''A' + \overline{D}^{\mathrm{T}}\overline{C}$, we introduce an $N \times (N+1)$ matrix Q'' and an $(N+1) \times N$ matrix Q':

$$Q''_{nm} = A''_{nm} \qquad Q''_{n0} = -\mathrm{i}\overline{D}_n \qquad Q'_{nm} = A'_{nm} \qquad Q'_{n0} = \mathrm{i}\overline{C}_n \tag{A7}$$

with $n, m = 1, 2, \cdots, N$. We thus have

$$\det(I + Q''Q') = 1 + \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} \sum_{0 \leqslant m_1 < m_2 < \cdots < m_r \leqslant N} \times Q''(n_1, n_2, \cdots, n_r; m_1, m_2, \cdots, m_r) Q'(m_1, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r). \tag{A8}$$

The sum is obviously decomposed into two parts: one is extended to $m_1 = 0$, the other to $m_1 \geqslant 1$. Except for the sum extended to $m_1 = 0$, (A8) is just (A1), because of (A6) and (A7). Hence we have

$$\det(I + Q''Q') - \det(I + A''A') = \sum_{r=1}^{N} \sum_{1 \leqslant n_1 < n_2 < \cdots < n_r \leqslant N} \sum_{1 \leqslant m_2 < \cdots < m_r \leqslant N} \times Q''(n_1, n_2, \cdots, n_r; 0, m_2, \cdots, m_r) Q'(0, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) \tag{A9}$$

where

$$Q''(n_1, n_2, \cdots, n_r; 0, m_2, \cdots, m_r) Q'(0, m_2, \cdots, m_r; n_1, n_2, \cdots, n_r) = (-1)^{r+1} \times \prod_n \prod_m \overline{D}_n \overline{C'}_n D_m C''_m \bar{\kappa}_n^{-1} \kappa_m (\bar{\kappa}_n - \kappa_m)^{-2} \prod_{n<n'} \prod_{m<m'} (\bar{\kappa}_n - \bar{\kappa}_{n'})^2 (\kappa_m - \kappa_{m'})^2 \tag{A10}$$

and

$$n, n' \in \{n_1, n_2, \cdots, n_r\} \qquad m, m' \in \{m_2, \cdots, m_r\}. \tag{A11}$$

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